Linear entropy as an entanglement measure in two-fermion systems

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We describe an efficient theoretical criterion, suitable for indistinguishable particles to quantify the quantum correlations of any pure two-fermion state, based on the Slater rank concept. It represents the natural generalization of the linear entropy used to treat quantum entanglement in systems of non-identical particles. Such a criterion is here applied to an electron-electron scattering in a two-dimensional system in order to perform a quantitative evaluation of the entanglement dynamics for various spin configurations and to compare the linear entropy with alternative approaches. Our numerical results show the dependence of the entanglement evolution upon the initial state of the system and its spin components. The differences with previous analyses accomplished by using the von Neumann entropy are discussed. The evaluation of the entanglement dynamics in terms of the linear entropy results to be much less demanding from the computational point of view, not requiring the diagonalization of the density matrix.

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I. INTRODUCTION

The entanglement, possibly the most remarkable feature of quantum mechanics, represents a fundamental resource for quantum information processing [1, 2, 3], and the concept of bipartite and multipartite entanglement is nowadays well-stated for quantum systems composed of distinguishable constituents. For an entangled system it is impossible to factor its state in a product of independent states describing its parts. On the other

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hand the notion of entanglement is more controversial in systems of identical particles [4, 5, 6, 7, 8, 9, 10, 11]. The entanglement of such systems is investigated at present in many areas of physics, like quantum optics, quantum charge transport in semiconductor, ultracold gases [12, 13, 14, 15, 16, 17, 18]. Here the difficulties appear in the definition of a criterion able to classify and quantify the entanglement. They are mainly due to the exchange symmetry which requires the antisymmetrization or symmetrization of the quantum wavefunctions describing fermions or bosons, respectively.

Different methods to treat the quantum entanglement in systems of indistinguishable particles are present in literature. In the approach developed by Wiseman and Vaccaro [10] the entanglement of the particles is a sort of accessible entanglement, i.e. the maximum value of the entanglement that could be extracted from the system and placed in quantum registers, from which it could be used to perform quantum information processing. In the theory introduced by Zanardi the entanglement should be evaluated by using the density matrix in a mode-occupation represention and is based on the formal mapping of the Fock space into states of qubits [7]. The method proposed by Schliemann is based on the Slater rank of the state (i.e. the minimum number of Slater determinants needed to express it) as a counterpart of the Schmidt rank criterion usually adopted for distinguishable particles [4, 19, 20]. This latest approach has been recently reexamined by other authors, which, for the case of a two-particle pure state, have suggested to evaluate quantitatively the entanglement as the von Neumann entropy (vNE) of the one-particle reduced density matrix [9, 21, 22]. Here the quantum correlations due to symmetrization or antisymmetrization of the wavefunction do not represent a genuine manifestation of quantum entanglement [23]. Therefore from this point of view such a criterion results to be the natural generalization of the approach commonly used to treat quantum correlations in systems of distinguishable particles.

Following the basic concepts of the approach proposed by Schliemann [4] in this paper we discuss an entanglement criterion for two-fermion systems. It is still based on the analogous of the Schmidt decomposition theorem for the pure fermion state but requires the calculation of the linear entropy (LE) of the one-particle reduced density matrix. We intend to establish whether the LE can be considered a valid measure of the lack of knowledge about the quantum state describing the system, which should include not only the uncertainty due to the impossibility of attributing a definite state to each particle but also the amount of uncertainty deriving from the indistinguishability of the particles. In order to get a better

understanding of this criterion, we first apply it to study a simple prototype theoretical model, then we analyze a system of physical interest, namely a two-electron scattering event in a 2D semiconductor structure. The entanglement dynamics in such a system has been recently investigated in terms of the vNE [15, 24] and a comparison with previous results lead to the conclusion that the LE can be an efficient and still valid entanglement measure for binary collisions, as for other physical phenomena of interest in quantum-information processing where identical particles are involved.

The paper is organized as follows. In Sec. II we describe the main properties of the LE as an entanglement measure for two-fermion systems. In Sec. III we evaluate numerically the time evolution of the LE for a scattering event between a free propagating and a bound electron in a two-dimensional system considering different spin configurations. Conclusions are drawn in Sec. IV.

II. THE THEORETICAL CRITERION AND ITS EVALUATION IN A $2N$-MODES SYSTEM

In this section we introduce the entanglement criterion for two-fermion systems based on the concept of LE, usually applied to distinguishable particles. Furthermore we compare quantitatively such a criterion with the one based on the vNE in the case of a simple system with $2N$ degrees of freedom.

A pure state of two fermions can be written as [4, 19]:

$$|\Psi_F\rangle = \sum_{i,j}^{2N} \omega_{ij} a_i^\dagger a_j^\dagger |0\rangle \tag{1}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operators of the mode i satisfying the usual fermionic (anti)commutation rules $\{a_i, a_j^\dagger\} = \delta_{ij}$, and $|0\rangle$ is the vacuum state. ω_{ij} are the elements of a complex and antisymmetric $(2N \times 2N)$ matrix Ω where $2N$ is the total number of modes for each single particle, while the normalization condition is given by $\mathrm{Tr}[\Omega^\dagger\Omega] = 1$. The single-particle reduced density matrix ρ for the state $|\Psi_F\rangle$ can be computed from the two-particle density matrix $\rho_F = |\Psi_F\rangle\langle\Psi_F|$ and its elements are [9]:

$$\rho_{\mu\nu} = \frac{\mathrm{Tr}[\rho_F a_\nu^\dagger a_\mu]}{\mathrm{Tr}[\rho_F \sum_\mu a_\mu^\dagger a_\mu]} = (\Omega^\dagger\Omega)_{\nu\mu} \tag{2}$$

The eigenvalues of ρ are $|z_i|^2$, while the coefficients z_i stem from the Schmidt decomposition of $|\Psi_F\rangle$ in terms of Slater determinants [4]. Furthermore it should be noticed that the eigenvalues of the one-particle reduced density matrix are pairwise identical and therefore it holds $|z_{2k}|^2 = |z_{2k-1}|^2$ with $\quad 1 \leq k \leq N$. The number of coefficients z_k that are different from zero is the so-called Slater rank, which can be related to the entanglement as follows: a state with Slater rank equal to 1 (i.e. that can be written as a single Slater determinant) is non-entangled, a state with Slater rank greater than 1 is a linear combination of two or more Slater determinants, therefore it can be considered entangled.

Many alternative ways of defining a function apt at evaluating the lack of knowledge about a subsystem have been proposed in the literature among them the Tsallis entropy [25] generalizes the concept of the vNE, encompassing, among the others LE. It is defined as [26] :

$$\varepsilon_{_q} = \frac{1}{q-1}\mathrm{Tr}\{\rho - \rho^q\} \tag{3}$$

where q is a real, not necessary positive, number. In the case of q tending to 1 one obtains the well known vNE $\varepsilon_{_{vN}} = -\mathrm{Tr}\{\rho \ln \rho\}$ which satisfies some standard properties as concativity, additivity and sub-additivity, and which is acknowledged to be a good quantum correlation measure of a pure two-fermion state. When q is equal to 2 Eq. (3) reduces to $\varepsilon_{_L} = 1 - \mathrm{Tr}\rho^2$, that is the LE [27]. In this paper we focus on such a quantity as a measure of the entanglement in systems of indistinguishable particles. Even if LE is not additive in the usual sense as shown in Ref.[27], it has some interesting properties so far not fully exploited. In fact it turns out to be extremely valuable for the application of numerical methods, as detailed in the following.

In terms of LE the quantum entanglement of the pure two-fermion state $|\Psi_F\rangle$ defined in Eq. (1) is given by

$$\varepsilon_{_L} = 1 - \sum_{i,j}^{2N} \left| \sum_{l}^{2N} \omega_{il}\omega_{lj}^{*} \right|^2 . \tag{4}$$

From the above expression we observe that the evaluation of the LE can be performed directly from the matrix Ω thus , in numerical calculations, the definition and allocation of the one-particle density matrix ρ are not required: therefore $\varepsilon_{_L}$ is much easier to calculate than the vNE since no diagonalization of the matrix ρ is needed. This aspect appears to be relevant since the complexity of many systems of physical interest practically prevents the diagonalization of the corresponding density matrix. Furthermore we note that Eq. (4) is

represention-independent since the trace is invariant with respect to unitary single-particle transformations.

By using the above trace operations we can also express $\varepsilon_{_L}$ in terms of the eigenvalues $|z_i|^2$ of the one-particle density matrix ρ in order to compare the expression of the LE with the one of the vNE. By taking into account the above mentioned property for $|z_i|^2$, we find for the former that

$$\varepsilon_{_L} = 1 - \sum_{k}^{N} 2|z_k|^4 \tag{5}$$

while, as shown in the literature [9, 21, 22], the latter can be written as

$$\varepsilon_{_{vN}} = \ln 2 - \sum_{k}^{N} 2|z_k|^2 \ln 2|z_k|^2. \tag{6}$$

We stress that Eq. (5) has been reported in oder to make more explicit the following discussion, but it is not employed in the numerical calculations where Eq. (4) is used instead, not requiring the calculations of the Ω eigenvalues. In spite of similarities between the two expressions for a two-fermion system, $\varepsilon_{_L}$ and $\varepsilon_{_{vN}}$ have a different dependence upon the coefficients z_k. Both Eqs. (5) and (6) attain their minimum value when the state $|\Psi_{_F}\rangle$ can be written in terms of a single Slater determinant. In this case we have $|z_1|^2 = \frac{1}{2}$ while all the other coefficients are zero and therefore $\varepsilon_{_L} = \frac{1}{2}$ and $\varepsilon_{_{vN}} = \ln 2$. The fact that the minimum value of the two measures is not 0, differently from what happens for the distinguishable particles case, is related to the unavoidable correlations due to the exchange symmetry. Since the quantum correlations related only to antisymmetrization of the state of two fermions cannot be used to violate Bell's inequality and are not a resource for quantum-information processing (as shown in previous works [21, 23]) a state with Slater rank equal to 1 can be considered as non-entangled . As a consequence we will assume that a value $\varepsilon_{_L} = \frac{1}{2}$ indicates a non-entangled state.

For a maximum correlated state it holds $|z_k|^2 = \frac{1}{2N}\ \forall\ k$ and $\varepsilon_{_L} = 1 - \frac{1}{2N}$. We note that in the case of a two-fermion system with a very large number $2N$ of modes, the maximum value of the LE tends asymptotically to 1 with a power law as for distinguishable particles.

In order to compare some properties of the LE and vNE here we shall analyze the two entanglement criteria for a simple $2N$-modes two fermion system in a state $|\chi\rangle$, that can be

expressed by Eq. (1), with the following coefficients of the antisymmetric matrix Ω:

$$\omega_{\rm ij} = \begin{cases} \sqrt{\frac{1+(N-1)(1-\alpha)^2}{2N}} & \text{for} \quad i=1, j=2 \\ \sqrt{\frac{\alpha(2-\alpha)}{2N}} & \text{for} \quad i=2k-1, j=2k \quad \text{with} \quad 2 \leq k \leq N \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where α is a real parameter ranging between 0 and 1. In particular we observe that for $\alpha = 0$ the elements of the matrix vanish except for ω_{12} which reduces to $\sqrt{\frac{1}{2}}$. In this case the state $|\chi\rangle$ can be set in terms of a single Slater determinant and therefore is non-entangled. On the other hand when $\alpha = 1$ the condition of maximum entanglement is reached, with all the non vanishing coefficients equal to $\sqrt{\frac{1}{2N}}$.

In the above model the one-particle reduced density matrix ρ is simply a diagonal matrix with eigenvalues ω_{12}^2 and $\omega_{2k-1,2k}^2$. This makes straightforward the calculation of the matrix ρ^2. Its trace, needed for the evaluation of the LE , can be written as function of the parameter α as:

$$\varepsilon_L^\chi = 1 - \frac{1}{2N} - \frac{(1-\alpha)^4(N-1)}{2N}. \tag{8}$$

As expected for $\alpha = 0$, ε_L^χ takes its minimum value $\frac{1}{2}$, while the maximum value of $1 - \frac{1}{2N}$ is reached for $\alpha = 1$. The expression of ε_L normalized to 1 reads

$$\tilde{\varepsilon}_L^\chi = 1 - (1-\alpha)^4. \tag{9}$$

The vNE can also be easily calculated and its normalized form is

$$\begin{aligned} \tilde{\varepsilon}_{vN}^\chi = \frac{-1}{N \ln N} \Bigg(& (N-1)\alpha(2-\alpha) \ln \frac{\alpha(2-\alpha)}{N} \\ + \quad & \big(1 + (1-\alpha)^2(N-1)\big) \ln \frac{1+(1-\alpha)^2(N-1)}{N} \Bigg). \end{aligned} \tag{10}$$

We note that the normalization of the vNE and of the LE allows us to compare quantitatively the two measures of the quantum entanglement. In Fig. 1 we report the entanglement of the system as a function of the real parameter α. Both curves get their minimum value 0 for $\alpha = 0$, both increase with α and reach 1 when α gets to 1. From the comparison we note that the LE is always greater than the vNE as a consequence of the different normalization procedures apart from the initial and final values when they coincide. For α tending to 1 (i.e. for $|\chi\rangle$ tending to the maximally entangled state) the two measures attain the same value as for the case of distinguishable particles [28]. For the sake of completeness we show the two non-normalized curves in the inset of Fig. 1.

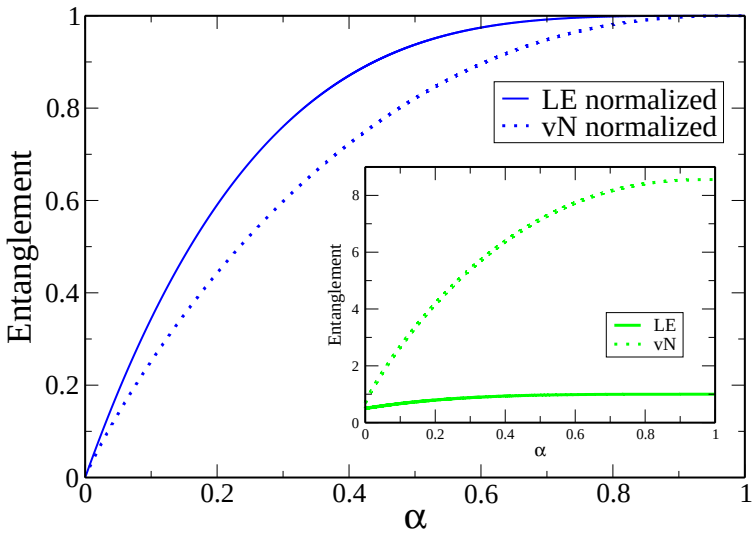

FIG. 1: (Color online) Comparison between the normalized values of the vNE and the LE as a function of the real parameter α in the $2N$ modes two-fermion system described in the text (for this calculation N has been taken equal to 2601). The inset displays the dependence of the two not normalized measures upon α.

III. LINEAR ENTROPY DYNAMICS IN A TWO-ELECTRON SCATTERING

A. The model

The dynamics of quantum entanglement has been investigated in various physical phenomena including, for example, ionization processes [29], binary collision events [15, 30, 31] and phonon-atom interaction [32]. Due to the increasing quest for quantum computing capable solid state devices, the study of the entanglement formation in a scattering event in semiconductor structures plays an important role. A numerical analysis of the entanglement dynamics in terms of vNE for a two-electron collision in two-dimensional semiconductor nanostructures (GaAs) has been recently presented in literature [15, 24] analyzing the evolution of the entanglement when an electron freely propagating interacts through a screened Coulomb potential with another electron bound to a specific site by a harmonic potential. In this section we intend to study the entanglement for such a model by using the criterion based on the LE described in the previous section.

The Hamiltonian describing the physical system can be written as:

$$H(\mathbf{r}_a, \mathbf{r}_b) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial \mathbf{r}_a^2} + \frac{\partial^2}{\partial \mathbf{r}_b^2}\right) + \frac{e^2}{\epsilon|\mathbf{r}_a - \mathbf{r}_b|} + \frac{1}{2}m\omega^2(\mathbf{r}_a - \mathbf{r}_0)^2 + \frac{1}{2}m\omega^2(\mathbf{r}_b - \mathbf{r}_0)^2 \quad (11)$$

where ϵ and m are the GaAs dielectric constant and effective mass, respectively, and $\mathbf{r}_0$ is the center of the harmonic potential, with energy-level spacing $\hbar\omega$. Spin-orbit effects have not been considered. At the initial time t_0 one of the two particles, namely the incoming electron, is represented by a minimum uncertainty wave-packet centered in $\mathbf{r}_0$(see Fig. 2a):

$$\psi(\mathbf{r}, t_0) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{4\sigma^2} + i\mathbf{k}\cdot\mathbf{r}\right) \tag{12}$$

where σ is the mean spatial dispersion, $k = \sqrt{2mE_k}/\hbar$ with m the effective mass of the carrier and E_k is the carrier kinetic energy. The bound electron is in the ground state of a two-dimensional harmonic oscillator

$$\phi(\mathbf{r}, t_0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega(\mathbf{r}-\mathbf{r}_1)^2}{2\hbar}\right) \tag{13}$$

where $\mathbf{r}_1$ is the center of the harmonic potential, with energy-level spacing $\hbar\omega$. The distance $|\mathbf{r}_1 - \mathbf{r}_0|$ is such that at the initial time t_0 the Coulomb energy is negligible. We stress that, as a consequence of the scattering, the state of the particle confined in the harmonic oscillator (HO) changes and that the two-particle wavefunction can be expressed as a single Slater determinant $\Psi(\mathbf{r}, \mathbf{r}') = \psi(\mathbf{r})\,\phi(\mathbf{r}') - \phi(\mathbf{r})\,\psi(\mathbf{r}')$, only at the initial time. In order to better analyze the state of the bound particle, Figs. 2b-d display the square modulus of the projection of the antisymmetrized two-particle wavefunction $\Psi(\mathbf{r}, \mathbf{r}')$ at time $t = 480$ fs on the first three eigenstates $\xi_n (n = 1, 2, 3)$ of the harmonic oscillator: $\gamma_n(\mathbf{r}) = |\int d\mathbf{r}' \xi_n(\mathbf{r}')\Psi(\mathbf{r}, \mathbf{r}')|^2$. Note that the spectral decomposition on the HO ground state at time $t = 0$ (Fig. 2a), representing the square modulus of the one-particle initial wavefunction of the free carrier, has been almost totally transmitted with no reflected part at $t = 480$ fs . The scattering event leaves the HO in a superposition of excited states, as can been seen in Figs. 2c and 2d and for higher energies the peaks of the function $\gamma_n(\mathbf{r})$ are closer to the center of the HO. This is due to the different energies of the outgoing particle as the bound particle is left in different HO excited states. Note that the Coulomb potential also creates spatial correlations between the two electrons.

We consider now the effect of different initial spin configurations on the evolution of the entanglement. In the first quantum state studied the two electrons have the same spin (spin up):

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\psi\,\phi\rangle - |\phi\,\psi\rangle\Big)|\uparrow\uparrow\rangle \tag{14}$$

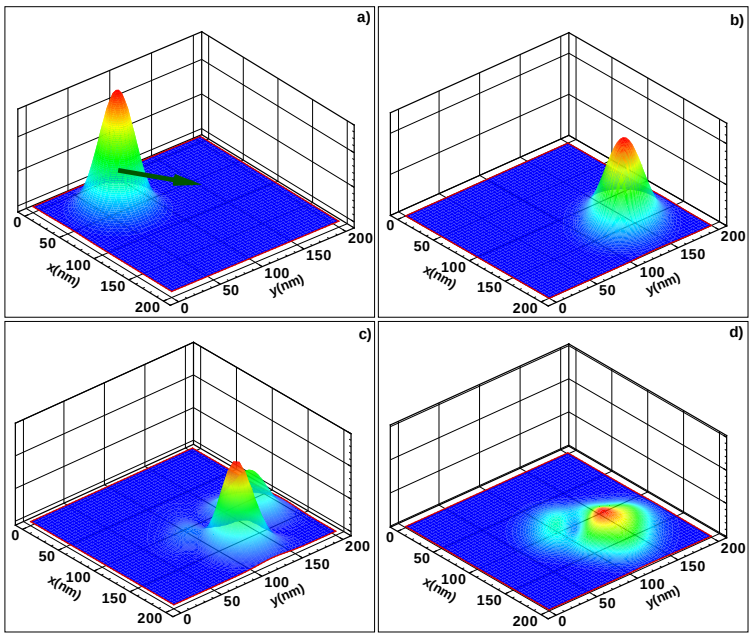

FIG. 2: (Color online) Square modulus of the projection of the antisymmetrized two-particle wavefunction on the first three energy eigenstates ξ_n of the harmonic oscillator centered in $\mathbf{r}_0$=(95 nm, 95 nm): $\gamma_n(\mathbf{r}) = |\int d\mathbf{r}'\xi_n(\mathbf{r}')\Psi(\mathbf{r},\mathbf{r}')|^2$. The two upper graphs show the projection on the ground state at two different times $t = 0$ (a) and $t = 480$ fs (b), while in the lower graphs the projections on first (c) and the second (d) excited state at time $t = 480$ fs are reported.

where the wavefunctions corresponding to the states $|\psi\rangle$ and $|\phi\rangle$ are of the type defined in Eqs. (12) and (13), respectively, and the ket $|\uparrow\rangle$ indicates a spin up state. The explicit form of the matrix Ω for the state $|\Psi\rangle$ can be obtained by discretizing the spatial coordinate $\mathbf{r}$ into a N^2 points grid (N points for each spatial dimension) as shown in Ref. [15]. We get for the $2N^2 \times 2N^2$ matrix Ω_ψ

$$\Omega_\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Omega_A & 0 \\ 0 & 0 \end{pmatrix}, \tag{15}$$

where Ω_A is the antisymmetric $N^2 \times N^2$ matrix whose elements read

$$\omega_{ij} = \psi(\mathbf{r}_i)\,\phi(\mathbf{r}_j) - \phi(\mathbf{r}_i)\,\psi(\mathbf{r}_j). \tag{16}$$

The second case considered is the one with two electrons having different spins, that cannot be factorized in a spin and a real space term. The form of the two-particle state is

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}}\Big(|\psi\,\phi\rangle|\uparrow\downarrow\rangle - |\phi\,\psi\rangle|\downarrow\uparrow\rangle\Big). \tag{17}$$

By applying a procedure analogous to the one used for $|\Psi\rangle$ and introducing an unitary transformation for the spin variables [15], we get

$$\Omega_\Upsilon = \frac{1}{2}\begin{pmatrix} \Omega_A & -\Omega_S \\ \Omega_S & -\Omega_A \end{pmatrix}. \tag{18}$$

where Ω_S is the symmetric counterpart of Ω_A.

The last two states considered still describe two electrons with different spins, but contrary to the previous case they can be factorized in a position term and in a spin term. We can identify the singlet spin state

$$|\Phi\rangle = \frac{1}{2}\Big(|\psi\,\phi\rangle + |\phi\,\psi\rangle\Big)\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big) \tag{19}$$

with

$$\Omega_\Phi = \frac{1}{2}\begin{pmatrix} 0 & -\Omega_S \\ \Omega_S & 0 \end{pmatrix}, \tag{20}$$

and the triplet spin state

$$|\Xi\rangle = \frac{1}{2}\Big(|\psi\,\phi\rangle - |\phi\,\psi\rangle\Big)\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big) \tag{21}$$

with

$$\Omega_\Xi = \frac{1}{2}\begin{pmatrix} \Omega_A & 0 \\ 0 & -\Omega_A \end{pmatrix}. \tag{22}$$

The LE of the triplet state can be easily obtained from the one of the same-spin state $|\Psi\rangle$ [15, 21]. In fact for the eigenvalues of the one-particle reduced density matrix of the state $|\Xi\rangle$ it holds

$$|z_i^\Xi|^2 = |z_{i+N^2}^\Xi|^2 = \frac{1}{2}|z_i^\Psi|^2 \quad \text{for} \quad 1 \le i \le N^2. \tag{23}$$

Therefore from the Eq. (5) its LE is given by

$$\varepsilon_L^\Xi = 1 - 2\sum_i^{N^2} |\frac{z_i^\Psi}{2}|^4 = \frac{1}{2}(1 + \varepsilon_L^\Psi). \tag{24}$$

The property expressed by the relation (24) is due to the fact that the triplet state, like the singlet state, cannot be factorized in a space term and a spin term and this property remains true during the time evolution. From Eq. (24) we observe that the minimum value of the LE of $|\Xi\rangle$ is $\frac{3}{4}$, greater than the value of $\frac{1}{2}$ corresponding to a non-entanglement condition for any two-fermion state according the criterion introduced in the previous section. Such

an offset is related to the lack of knowledge about the spin of the particles. The behavior shown by the triplet and singlet states is in agreement with the one obtained in previous works where the entanglement formation is evaluated in terms of the vNE for the case of one-dimensional and two-dimensional scattering [15, 24], as we show in the following.

B. Numerical results

In order to calculate the entanglement dynamics in the system described above, we solve numerically the time-dependent Schröndiger equation for the two-particle wavefunction considering as initial condition two electrons described by the wave-packets given in the Eqs. (12) and (13). In this way at each time step we have the two-particle wavefunction needed to define the matrix Ω. Finally from expression (4) and by using only the matrix elements ω_{ij} we can compute the time evolution of the entanglement in terms of the LE . We stress that such an approach is simple and does not require matrix diagonalization procedures which result to be very demanding from the point of view of the numerical calculation.

Figure 3 shows that at initial time the LE for the states $|\Psi\rangle$ and $|\Upsilon\rangle$ get its minimum value $\frac{1}{2}$. In fact the Coulomb energy is still negligible being the two wave-packets far enough, and the only quantum correlations present are due to the exchange symmetry. Therefore, as expected, $|\Psi\rangle$ and $|\Upsilon\rangle$ must be considered as initially non entangled. We observe that as the free carrier get closer to the center of the harmonic potential the quantum correlation builds up and the LE reaches a stationary value once the particles get far enough. Such a value depends upon the initial kinetic energy of the propagating carrier E_k: in particular it is higher for higher energies for both spin configurations, in good qualitative agreement with the previous results found for scattering events between two distinguishable particles [33, 34, 35].

In Fig. 4 the time evolution of the entanglement for the singlet $|\Phi\rangle$ and triplet $|\Xi\rangle$ spin states are presented. As expected, at the initial time the LE is $\frac{3}{4}$. This implies that $|\Phi\rangle$ and $|\Xi\rangle$ are initially entangled, being their Slater rank greater than 1. In fact from Eqs. (19) and (21) we observe that they cannot be put in terms of a single Slater determinant. Such a result confirms the correspondence between the LE and the Slater rank criterion.

In order to better compare the properties of the entanglement measure for an electron-electron scattering obtained using the LE and vNE, the time evolution of the entanglement,

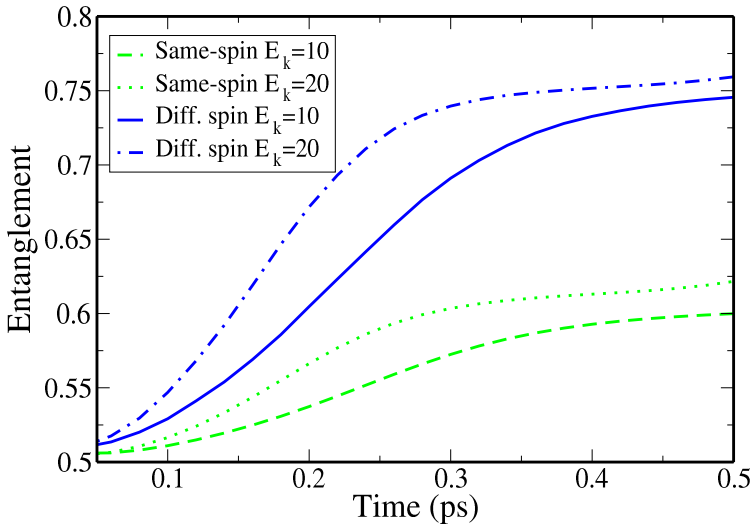

FIG. 3: (Color online) Entanglement as a function of the time for different initial states: same-spin state $|\Psi\rangle$ for two different initial energy values of the incoming electron $E_k = 10$ meV (dashed line) and 20 meV (dotted line); the state with electron having different spin $|\Upsilon\rangle$ for $E_k = 10$ meV (dot-dashed line) and 20 meV (solid line). The harmonic oscillator energy is $\hbar\omega = 2$ meV.

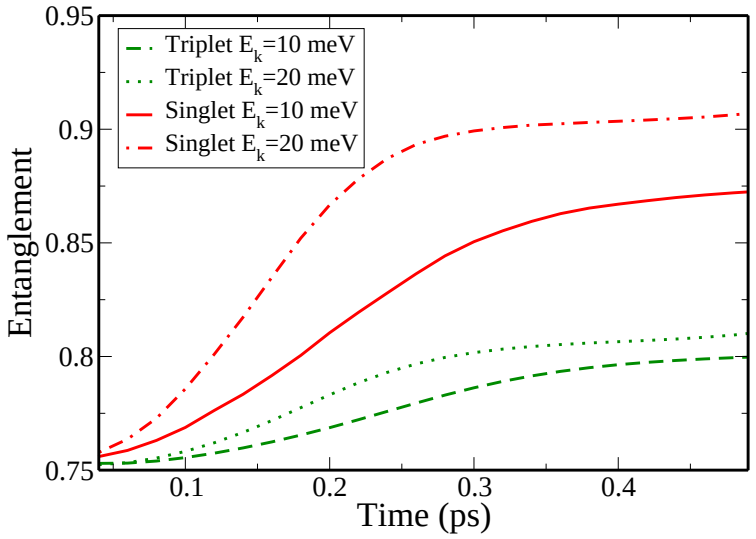

FIG. 4: (Color online) Entanglement as a function of the time: triplet spin state $|\Xi\rangle$ for two different initial energy values of the incoming electron $E_k = 10$ meV (dashed line) and 20 meV (dotted line) and singlet spin state $|\Phi\rangle$ for $E_k = 10$ meV (dot-dashed line) and 20 meV (solid line). The harmonic oscillator energy is $\hbar\omega = 2$ meV.

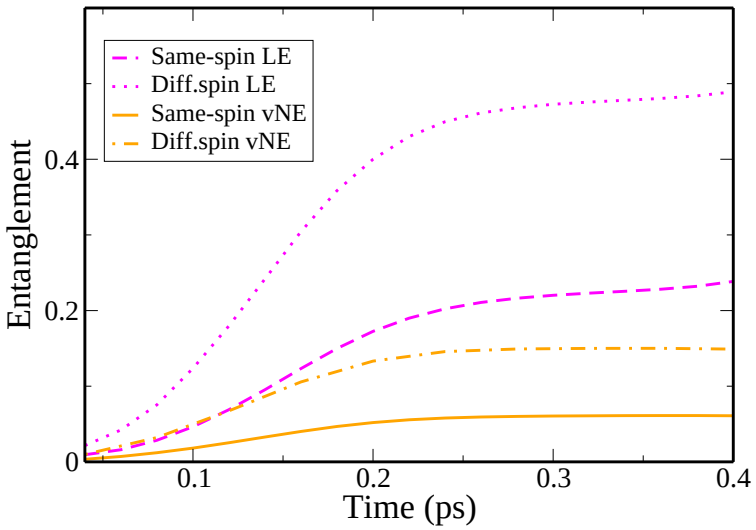

FIG. 5: (Color online) Two measures of the entanglement as a function of the time. The time evolution of the normalized LE for same spin state $|\Psi\rangle$ (dashed line) and for the state with electrons having different spin $|\Upsilon\rangle$ (dotted line), is compared with the time evolution of the normalized vNE for the same states: $|\Psi\rangle$ (solid line) and $|\Upsilon\rangle$ (dash-dotted line). The initial kinetic energy of the incoming electron is 30 meV while the harmonic oscillator energy is $\hbar\omega = 2$ meV.

evaluated according to the two criteria, is presented in Fig. 5 for the states $|\Psi\rangle$ and $|\Upsilon\rangle$. As for the case of the theoretical model studied in the previous section, here we have normalized the two measures in order to compare them quantitatively. At initial time both of them are zero since no quantum correlation is initially present apart from the one related to the exchange symmetry. Then, at increasing times, for a given state the LE is always greater than the vNE. Nevertheless the time of the entanglement formation, defined as the time at which the entanglement reaches its stationary value, is the same for both measures. This behavior can be ascribed to the fact that, as indicated by the Eqs. (5) and (6), both the LE and vNE can be expressed as function of $|z_k|^2$, the eigenvalues of the one-particle reduced density matrix, which can be assumed weakly time-dependent for large times.

IV. CONCLUSION

In the last years the notion of the entanglement for systems of identical particles has been widely discussed: various approaches have been proposed in literature each having advantages and drawbacks [4, 7, 8, 9, 10]. In this paper we have analyzed one possible theoretical

criterion to quantify the quantum correlations appearing in a two-fermion systems. It can be considered the generalization of the LE, usually adopted in the context of distinguishable particles. Such a criterion is based on the fermionic analogous of the Schmidt decomposition theorem and uses the one-particle reduced density matrix. In particular our analysis shows that LE permits to determine whether the uncertainty concerning the states derives only from the indistinguishability of the particles or it is a genuine manifestation of the entanglement. This aspect is crucial since, as shown in previous works, the quantum correlation due to the exchange symmetry does not represent a good resource for quantum information processing[21, 23].

The criterion proposed here appears to be closely related to the one involving the evaluation of the vNE of the reduced statistical operator [9, 21, 22]. To compare the two criteria and analyze their properties we have quantified the entanglement according to the two measures in a simple theoretical model describing a two-fermion system with $2N$ degrees of freedom. We found that the LE is greater than the vNE, but the two measures give the same value when the quantum state of the system tends to the maximally entangled one in agreement with results obtained for distinguishable particles [28].

Furthermore we have used the LE to quantify the time evolution of quantum correlations in a numerically simulated electron-electron collision event. In agreement with the previous analyses obtained by using the vNE [15, 16], our numerical results show that the entanglement dynamics depends on the spin components of the states even if the Hamiltonian does not include spin-terms. Moreover also in this case the triplet and singlet spin states are initially entangled and such an entanglement can be ascribed to the lack on knowledge about the spin state of a particle in a specific real-space state. For our two-electron model the values of the entanglement obtained by using the LE results to be higher than the ones obtained by using the vNE in agreement with what found analytically for a simple model of a $2N$-mode two-fermion system. Most notably we found that the time of the entanglement formation is the same for the two measures.

Finally we note that the calculation of the entanglement in terms of the LE is easier and computationally much faster than the one performed by means of the vNE since no diagonalization of the one-particle reduced density matrix is required. Therefore we believe that the LE is an useful correlation measure for a two-fermion system and its application turns out to be very helpful to investigate the entanglement dynamics for those physical

systems with a very large number of degrees of freedom, whose complexity does not allow the diagonalization of the reduced statistical operators through numerical procedures.

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